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CONDITIONAL OBJECTS AND THE MODELING OF UNCERTAINTIES

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Abstract

This paper proposes a qualitative approach to conditioning which can be used in the modeling of uncertainties, as for example, in the combination of evidence problems that arise in probabilistic or AI contexts. The resulting measure-free conditional objects are shown to be both compatible with, and to establish, new insights in the structure of ordinary conditional probabilities. In addition, explicit relations are developed between conditional objects and the often mistakenly equated standard logical implication operators. Extensions to other conditional entities, including fuzzy sets, are also outlined, as a special case of the main thesis of the paper: *that conditioning in any context can be identified as simply the inverse of the transform representing conjunction.*

Keywords: Conditional object, conditional probability, conditional fuzzy set, logical implication, inverse transform, uncertainty modeling.

1. INTRODUCTION

A basic problem in designing intelligent systems for AI is the production of feasible inference engines. This kind of inference involves essentially some type of logic. At the simplest level, for deterministic and classical two-valued logical systems, ordinary modus ponens is the universal inference engine. When probabilistic information is present, usually some form of bayesian updating technique is used as the engine, based on conditional probabilities and their associated calculus or logic. On the other hand, if information is in linguistic form or vague or only partially specified, then the technique of fuzzy logic is used as the inference engine. Even more generally, situations can arise where both types of uncertainty may be present, such as in a military scenario where evidence of an unknown target appears in the form of both descriptive narratives from experts in the field and statistical data obtained from sensor systems. (See Goodman and Nguyen [1] for a comprehensive treatment of this situation, where fuzzy and probabilistic modeling are unified and related.)

Regardless of the situation, information or evidence is, in general, in the form of at least implicitly stated conditional propositions or well-formed formulas. All of the above-mentioned approaches use some kind of semantic/numerical evaluation of these conditional propositions, whether they are in the form of incoming raw data or a priori knowledge, in conjunction with an associated logic, to define the appropriate inference engine.

A common method for directly modeling conditional statements is the use of the logical implication operator. (For a typical example of such identifications see, e.g., Watanabe, [2], Chapt. 7.3, pp. 333-347.) Generally, such implication is interpreted linguistically as "if b then a," symbolically, $b \Rightarrow a$, which in classical logic is simply "not b or a," i.e., $b' \vee a$ while in multivalued logics $b' \vee a$ or other formulations may be used to interpret implication [3]. In particular, when probability logic is used -- probability logic being, as is well known, a non-truth functional — aleph_1-valued logic (again, see [3]) — implication is commonly interpreted as "a given b," as distinct from "if b then a," which also can be evaluated -- usually as in classical logic as "not b or a." On the other hand, in common probability usage, the two concepts are often assumed to be the same and so used. However, it is surprising how few non-logicians, including probabilists, are aware that this common identification — called in logic "Stalnaker's Thesis" ([4], [5]) — leads to "triviality" results: It has been shown by P. Calabrese [6], [7] that no boolean binary function, including the usual classical logical interpretation, for implication can be identified in general with the formal entity $(a|b)$ within the standard conditional probability evaluation $p(a|b)$, with a similar, independent proof by D. Lewis [8] concentrating on only the identification of the classical logical implication operator with probabilistic conditioning. Later refinements were established by W. Rehder [9], with modifications and interpretations made by B.C. Van Fraasen [10], E.W. Adams [11], Chapt. 1, and D. Nute [12], where a survey of various proposed "conditional" logics is given. Additional discussions of the problem are presented by A. Appiah [13], Chapters 9-11 and I.R. Goodman and H.T. Nguyen [14].

As a simple illustration of the above, consider the following easily proven inequality (initially pointed out by Calabrese [7], with a simplified proof here):

Let $a, b \in \Omega$, a boolean ring with $p : \Omega \to [0,1]$ a probability measure, where $p(\cdot|\cdot\cdot)$ is the usual conditional probability operation and where the classical logical implication operation $\Rightarrow$ is defined by

$$(b \Rightarrow a) \overset{d}{=} b' \vee a \,. \tag{1.1}$$

Then

$$\begin{aligned} p(b \Rightarrow a) &= 1 - p(b \cdot a') = p(a|b) + p(a'|b) - p(a'|b) \cdot p(b) \\ &= p(a|b) + p(b') \cdot p(a'|b) \\ &\geq p(a|b) \,, \end{aligned} \tag{1.2}$$

with strict inequality holding in general!

In order to clarify the above situation, it is necessary to provide a clear and consistent concept of what $(a|b)$ really means and how it relates to $b \Rightarrow a$. It is also obvious (backed by an extensive literature search) that currently there are relatively few interpretations of $(a|b)$ without reference to probability. Among the exceptions should be noted Domotor [15] and Calabrese [7]. Domotor's initial idea that $(a|b)$ should be interpreted as a coset within the quotient boolean ring Ω/b' was used in a modified form by Calabrese. Domotor used such qualitative or measure-free conditional objects to aid in developing a general qualitative theory of probability compatible with preference ordering and subjective probability. However, little attention was paid to the development of explicit operations and relations between conditional entities having <u>different</u> antecedents. (See especially his comments on page 22 of [15].) Calabrese [7] filled this vacuum by producing an extensive calculus of relations and operations involving such forms.

This paper owes its genesis to the efforts of Calabrese and to a certain degree is based on an extension and modification of his earlier ideas.

The use of conditional object relations can be simply illustrated by the following. If we let

$b \overset{d}{=}$ " temperature $\leq 25°$ today"

$a \overset{d}{=}$ "snow will fall today,"

$c \overset{d}{=}$ "$20° \leq$ temperature $\leq 35°$ today"

$d \overset{d}{=}$ "snow or sleet will fall today,"

with all associated required probabilities known, then it may be desirable to compute $p((a|b) \cdot (c|d)')$ or perhaps $p((a|b) \vee (c|d))$, etc. All of those computations require an interpretation of what $(a|b)$ and $(c|d)$ actually mean and how to combine them first within the probability operator domain. Thus, if such "conditional objects" could be successfully defined and a calculus of operations developed, then a wide range of problems involving combination of evidence could be addressed within a formal language format, prior to semantic evaluations. Such measure-free conditional objects should be compatible with conditional probabilities and conditional fuzzy sets, i.e., when measures are assigned or more generally when semantic evaluations are taken.

In the next section (2), an outline of a theory of conditional objects is presented. As defined, a conditional object is not an element of the original base space or logic, but rather represents a <u>subset</u> of the logic, i.e., lies in the power class of the logic and hence <u>is at</u> a higher level then any unconditional or ordinary object or element of the space. The mathematical approach taken here to defining conditional objects is algebraic in nature, arising from the <u>identification of conditioning with the inverse of the function representing conjunction</u>. In Sections 2 and 3 it is shown that conditional objects for a boolean ring include the original elements of the boolean ring as special cases, and are the same as principal ideal cosets of the ring, each having as a maximal element, a logical implication. Also, in Section 3, a number of relations involving conditional objects is presented. In Section 4, further concepts are introduced, including iterated, or higher level, conditioning and a technique for best approximating (measure-free) arbitrary entities by conditional objects, with applications to functional extensions, and in particular, arithmetic operations, as well as to higher order conditional objects. In Section 5, conditional probability measures are briefly investigated.

2. CONDITIONING IDENTIFIED AS AN INVERSE CONJUNCTION OPERATION.

If it is reasonable to identify conditioning as simply a special functional inverse operation, the very extensive area of equation theory and operator inversion techniques over rings and more general structures are potentially available for use in investigating and developing properties of conditional forms.

Our thesis is based on the premise that conditioning in its broadest sense is identifiable with the category theory concept of the substitution of an arrow into a relation, which is defined as the action of the contravariant subobject functor Sub : SET → PREORD on an arrow (or function), evaluated at some relation ([1], Chapt. 2). For the problems we are interested in, this reduces to ordinary functional inverses. Specifically, to define "a given b," symbolically, from now on, $(a|b)$, where a and b are sentences, or ordinary sets, or fuzzy sets, or any other entities in some fixed universe of discourse Ω, consider first the mapping $f_b : \Omega \to \Omega$, defined by

$$f_b(x) \overset{d}{=} x*b \ , \tag{2.1}$$

where $* : \Omega \times \Omega \to \Omega$ is a conjunction ("and") operation such as ordinary intersection, when Ω is a collection of sets and fuzzy intersection defined through t-norms, e.g., when Ω is a collection of fuzzy sets. In turn, there is the associated inverse mapping $f_b^{-1} : \Omega \to \mathcal{P}(\Omega)$, where $\mathcal{P}(\Omega)$ denotes the power class of Ω and, where as usual, the inverse image

$$f_b^{-1}(a) \overset{d}{=} \{x | x \in \Omega \ \text{ and } \ f_b(x) = a\} \ . \tag{2.2}$$

Thus, finally, if we define

$$(a|b) \overset{d}{=} f_b^{-1}(a*b) \ , \tag{2.3}$$

"a given b" can be interpreted as representing any element x of Ω which when conjoined with b yields back $a*b$, which it should be noted, is always in the range of f_b. It follows immediately, extending f_b to $f_b : \mathcal{P}(\Omega) \to \mathcal{P}(\Omega)$ in the usual component-wise class sense i.e., for any $A \subseteq \Omega$,

$$b*A = f_b(A) \overset{d}{=} \{f_b(x) | x \in A\} \subseteq \Omega \ , \tag{2.4}$$

that

$$f_b(f_b^{-1}(a*b)) = b*(a|b) = a*b \ . \tag{2.5}$$

Thus, if $a \subseteq b$ with a, b sets in Ω, a boolean ring of sets, and $*$ represents ordinary set intersection $\cap$, and $p : \Omega \to [0,1]$ is some probability measure, then the usual definition of conditional probability is determined through a homomorphism of the relation in the right side of (2.5), where, formally,

$$p(\cap) \overset{d}{=} \cdot \text{ (product) },\tag{2.6}$$

resulting in

$$p(b) \cdot p(a|b) = p(a) .\tag{2.7}$$

Note that for the above special case for ordinary sets, f_b is a <u>linear</u> function relative to set unions $\cup$ or set symmetric differences Δ. However, in general, linearity need not hold, such as when Ω represents the class of all fuzzy subsets of some set and where $*$ is evaluated through some general intersection (conjunction) operation, typically being a t-norm or copula, and where union (disjunction) is described by some t-conorm or co-copula. In this case, Ω can retain certain ring properties in general, such as closure, commutativity, often, associativity, and even perhaps idempotence, for the conjunction and disjunction operations, the latter holding only when min and max are chosen to represent conjunction and disjunction respectively.

Unfortunately, because of space limitations, we will not pursue any further the concept of conditional fuzzy sets as derived from the viewpoint in (2.3). Future papers will cover this important omission.

Going in a direction more structured than fuzzy sets, a number of results have been extended especially most of Section 3, where a calculus of conditional objects is displayed: with both Ω assumed boolean and, more generally, with Ω being any commutative Von Neumann regular ring with unity; and to even more general structures (see [14]). Again, space preludes further discussion.

From now on, suppose Ω represents a boolean ring of ordinary subsets of a given base space, with the usual boolean set operations $\cap$ (intersection), Δ (symmetric set difference), $\cup$ (union), C (complement), and partial order relation $\subseteq$ (subset), or equivalently (isomorphically, order preserving), via Stone's Representation Theorem, suppose Ω is an (algebraic) boolean ring of propositions or sentences, etc., with corresponding operations $\cdot$ or $\wedge$ (conjunction or ring multiplication), + (ring addition), $\vee$ (disjunction), $(\ )'$ (negation or complement) and the partial order $\leq$ over Ω. For convenience, we will usually use the latter interpretation, with the former reserved for certain applications such as for the extension of ordinary arithmetic operations over the real numbers to real conditional sets.

The above basic assumption, implies, using (2.5), that conditional objects satisfy the characterizing implicit relation

$$b\cdot(a|b) = a\cdot b = a\cdot(b|a)\tag{2.8}$$

and that explicitly,

$$\begin{aligned}(a|b) = (a\cdot b|b) &= \{x|x \in \Omega \text{ and } x\cdot b = ab\}\\ &= a + \Omega\cdot b' = a\cdot b\vee\Omega\cdot b' ,\end{aligned}\tag{2.9}$$

where

$$\Omega\cdot b' = \{x\cdot b'|x \in \Omega\} = \{y|y \in \Omega \text{ and } y \leq b'\}\tag{2.10}$$

is the principal (product) ideal in Ω generated by b'.

It follows, for any $a, b, c, d \in \Omega$,

$$(a|b) = (c|d) \text{ iff } (ab|b) = (cd|d) \text{ iff } ab = cd \text{ and } b = d\ , \qquad (2.11)$$

and the relation in (2.10) is consistent relative to all substitutions.

Note also that when $b = 1$ (the multiplicative identity element for Ω or equivalently the entire base space X, when $\Omega \subseteq \mathcal{P}(X)$ is interpreted as a collection of sets),

$$(a|1) = a \in \Omega\ , \qquad (2.12)$$

while for $b = 0$ (the additive identity element or null set $\emptyset$, when $\Omega \subseteq \mathcal{P}(X)$)

$$(a|0) = \Omega\ . \qquad (2.13)$$

From now on, denote $\tilde{\Omega}$ as the class of all conditional objects $(a|b)$ ($\subseteq \Omega$), noting that (2.12) shows $\tilde{\Omega}$ extends Ω, i.e.,

$$\Omega \subseteq \tilde{\Omega} \subseteq \mathcal{P}(\Omega)\ , \qquad (2.14)$$

with, in general, strict subclass inclusion holds throughout (2.14). Indeed, since obviously

$$\tilde{\Omega} = \bigcup_{b \in \Omega} (\Omega/b)\ , \qquad (2.15)$$

where $\cup$ represents the class union operator and for each $b \in \Omega$, Ω/b is the quotient boolean ring

$$\Omega/b = \{a + \Omega\cdot b = (a|b'):a \in \Omega\}\ , \qquad (2.16)$$

it follows that

$$\text{card}(\tilde{\Omega}) = \sum_{b \in \Omega} \text{card}(\Omega/b) = \sum_{b \in \Omega} \text{card}(\Omega\cdot b)\ . \qquad (2.17)$$

Thus, if $\Omega = \mathcal{P}(X)$ for some finite X, with $\text{card}(X) = m$,

$$\text{card}(\tilde{\Omega}) = \sum_{t=0}^{\infty} \binom{m}{t} \cdot 2^t = 3^m < 2^{2^m} = \text{card}(\mathcal{P}(\Omega))\ . \qquad (2.18)$$

As mentioned, Calabrese earlier introduced measure-free conditional objects and proposed definitions for operators upon these objects which extend the ordinary boolean ones upon Ω.

First, consider Calabrese' definition of a conditional object, denoted with a subscript $\circ$ to distinguish from that presented here. It can be shown, that despite the rather different approach, Calabrese' definition is indeed equivalent to ours. More specifically, note first that any $L \subseteq \Omega$, boolean, is a <u>dual ideal</u> iff, for any $p, q \in L$, $s \in \Omega$, $p\cdot q$, $p \vee s \in L$. It follows immediately that for any (ordinary) ideal $I \subseteq \Omega$, $I' = \{p':p \in I\}$ is a dual ideal, and conversely, any dual ideal $L = I'$, for $I \overset{d}{=} L'$, an ideal of

Ω. In particular, any principal dual ideal $\Omega \vee b$, for any $b \in \Omega$, can be expressed as

$$\Omega \vee b = \{s \vee b : s \in \Omega\} = \{x : x \in \Omega,\ x \geq b\} = (\Omega \cdot b')' , \tag{2.19}$$

and conversely, one can express (ordinary) principal ideals as complements of corresponding principal dual ideals. Then it follows that, in effect, Calabrese' definition is, for any dual ideal L of Ω and any $a \in \Omega$,

$$(a|L)_o \overset{d}{=} \{x : x \in \Omega \text{ and there exists } r \in L \text{ such that } x \cdot r = a \cdot r\} . \tag{2.20}$$

(See [7], definition 3.3.2.)
Next, let, for a and L as above,

$$(a|L)_1 \overset{d}{=} \{y : y = x \cdot r' \vee a \cdot r,\ x \in \Omega,\ r \in L\} . \tag{2.21}$$

It follows readily that

$$(a|L)_o = (a|L)_1 . \tag{2.22}$$

In turn, for any such y as in (2.21), note the identity

$$y = x \cdot r' + a \cdot r = x \cdot r' + a(r' + 1) = (x + a) \cdot r' + a . \tag{2.23}$$

Thus, for any $a \in \Omega$ and dual ideal $L \subseteq \Omega$, (2.22) and (2.23) imply

$$\begin{aligned} (a|L)_o &= \{y : y = (x + a) \cdot r' + a,\ x \in \Omega,\ r \in L\} \\ &= (\Omega + a) \cdot L' + a \\ &= \Omega \cdot L' + a \\ &= L' + a , \end{aligned} \tag{2.24}$$

noting from previous remarks that L' is an ordinary ideal.

In particular, letting $L = \Omega \vee b$ in (2.24), it follows that for any a, $b \in \Omega$,

$$(a|b)_o \overset{d}{=} (a|\Omega \vee b)_o = (\Omega \vee b)' + a = \Omega \cdot b' + a = (a|b) , \tag{2.25}$$

using the basic relation in (2.9).

In a related direction, the following inclusion relation is easily verified for any ideal $I \subseteq \Omega$ and any $a \in \Omega$:

$$I + a \supseteq I' \cdot a,\ I \vee a . \tag{2.26}$$

Thus, by specializing I in (2.26) to principal ideal $\Omega \cdot b'$, it follows that the characterizing relation (2.8) for conditional objects will also be satisfied by the conjunction dual "cosets" $(\Omega \cdot b')' \cdot a = (\Omega \vee b) \cdot a$ and disjunction "cosets" $\Omega \cdot b' \vee a$. Thus, it might be tempting to use any of these classes of entities as possible alternative definitions for conditional objects. But these "cosets" are not true cosets of Ω, since for either type, the corresponding equivalence class property relative to all $a \in \Omega$ (b fixed) fails, due to a lack of symmetry, or equivalently, neither type of collection of "cosets" actually forms a disjoint partitioning of Ω.

Next, let us consider operators upon conditional objects which could possibly extend the notion of familiar operations upon ordinary (unconditional) objects in Ω, such as the usual boolean operations. Thus, if we let $f : \Omega^n \to \Omega$ be any n-ary operation, where $\Omega^n = \Omega \times \cdots \times \Omega$ is the usual cartesian n-product, $n \geq 1$, we can also let $f : \mathcal{P}(\Omega)^n \to \mathcal{P}(\Omega)$ be the usual component-wise class extension of f (as, e.g., in (2.4)). Thus, using (2.14), we can restrict the domain of the extension to $f : \tilde{\Omega}^n \to \mathcal{P}(\Omega)$ and inquire whether f is a legitimate operator over $\tilde{\Omega}^n$, i.e., if f yields closure, i.e., if $rng(f) \subseteq \tilde{\Omega}$ so that we can write $f : \tilde{\Omega}^n \to \tilde{\Omega}$ as an extension of $f : \Omega^n \to \Omega$.

The answer to this question is in the affirmative for all boolean operations acting upon conditional objects as defined here, with the results summarized in Theorem 3.1 of the next section. On the other hand, this is not necessarily the case for other operations, such as arithmetic ones. (See Section 4.) On the other hand, one can simply define, in some ad hoc manner, an extension of an operator $f : \Omega^n \to \Omega$ to $f_1 : \tilde{\Omega}^n \to \tilde{\Omega}$, or to $f_1 : \tilde{\Omega}_o^n \to \tilde{\Omega}_o$, which need not coincide with the component-wise class extension used throughout this paper. But care must be exercised that f_1 is well-defined. (It is easily verified that all component-wise class extensions relative to $\tilde{\Omega}^n$ are well-defined relative to all substitutions and equality for conditional objects from $\tilde{\Omega}$.)

In [7], Calabrese proposed extensions for the common boolean operators $\cdot$, V, and $(\ )'$ from Ω to $\tilde{\Omega}$.

Specifically, the extensions are defined, using again a subscript o to distinguish Calabrese' approach from the one here: for any $a, b, c, d \in \Omega$,

$$(a|b)'_o \overset{d}{=} (a'|b) , \tag{2.27}$$

whence by (3.1), Calabrese' definition for the extension of complement to conditional objects coincides with that derived here. On the other hand,

$$(a|b) \vee_o (c|d) \overset{d}{=} (a\cdot b \vee c\cdot d|b \vee d) , \tag{2.28}$$

when compared with the corresponding extension for disjunction to conditional objects established here in (3.3), shows that the two extensions in general differ considerably in their antecedents. Similarly, Calabrese' proposal for the extension of conjunction of conditional objects, as a DeMorgan dual of disjunction,

$$(a|b) \cdot_o (c|d) \overset{d}{=} ((a\vee b')\cdot(c\vee d')|b \vee d) \tag{2.29}$$

is quite distinct in form from the conjunction extension here, shown in (3.4).

In any case, it can be readily shown that all of the above proposed operator extensions are well-defined. With all due credit extended to the pioneering work in [7], the thrust of the approach taken in this paper to conditional objects, rather than appealing to intuition or analogues with other concepts (such as with logical implication), is to derive from first principles as many results as possible. One justification for considering the basic component-wise (or power class) extension of operators over Ω^n to $\tilde{\Omega}^n$, is that these extensions can be shown to be well-defined and to extend in a natural sense corresponding coset operations for fixed common antecedents of conditional objects. In turn, these properties, together with a partial order extension to $\tilde{\Omega}$, can be shown to lead a number of interesting and useful properties, as will be demonstrated in the following three sections.

3. DEVELOPMENT OF A CALCULUS OF OPERATIONS FOR CONDITIONAL OBJECTS.

In order for conditional objects to be applicable, we need to develop a calculus of operations for these entities. The results are summarized in the ensuing theorems, corollaries and remarks. By limitation of space, most proofs have been omitted. (See [14] for all relevant proofs and additional properties.) Relative to the discussion near the conclusion of the last section, as stated there, by utilizing the basic component-wise class extensions of operators over Ω^n, one can obtain legitimate extensions to $\tilde{\Omega}^n$ for at least the class of all boolean operators and happily these extensions have simple computable properties. Thus:

Theorem 3.1.

The boolean operators $(\ )'$, $+$, $\cdot$, $\vee$ are all well-defined and closed relative to $\tilde{\Omega}$. In addition, the relations are given for all $a, b, c, d \in \Omega$,

$$(a|b)' = (a'|b) = (a'b|b), \tag{3.1}$$

$$(a|b) + (c|d) = (a + c|bd) = (ab + cd|bd), \tag{3.2}$$

$$\begin{aligned}(a|b) \vee (c|d) &= (a \vee c|ab \vee cd \vee bd)\\ &= (ab \vee cd|ab \vee cd \vee bd)\,,\end{aligned} \tag{3.3}$$

$$\begin{aligned}(a|b) \cdot (c|d) &= (a\cdot c|a'b \vee c'd \vee bd)\\ &= (abcd|a'b \vee c'd \vee bd)\,.\end{aligned} \tag{3.4}$$

$(\ )'$ is involutive and since $+$, $(\ )'$, $\vee$ over $\tilde{\Omega}^n$ are easily shown to be associative and commutative, they are extendable to any finite number of arguments $(a_i|b_i)$, $i = 1, \cdots n$, whence

$$(a_1|b_1) + \cdots + (a_n|b_n) = (a_1 + \cdots + a_n|b_1 \cdots b_n)\,, \tag{3.5}$$

$$(a_1|b_1) \vee\cdots\vee (a_n|b_n) = (a_1 \vee\cdots\vee a_n|a_1b_1 \vee\cdots\vee a_nb_n \vee b_1\cdots b_n)\,, \tag{3.6}$$

$$(a_1|b_1) \cdots (a_n|b_n) = (a_1 \cdots a_n|a_1'b_1 \vee\cdots\vee a_n'b_n \vee b_1 \cdots b_n)\,. \tag{3.7}$$

□

Remark.

As mentioned before, the relations in Theorem 3.1 can be extended. For example, if $I, J \subseteq \tilde{\Omega}$ are any (product) ideals of Ω and if $\alpha, \beta \in \Omega$, then

$$(I + \alpha) + (J + \beta) = (I + J) + (\alpha + \beta) \tag{3.8}$$

$$(I + \alpha) \cdot (J + \beta) = I \cdot J + J\alpha + I \cdot \beta + \alpha \cdot \beta \ , \tag{3.9}$$

$$(I + \alpha) \vee (J + \beta) = I \cdot J + J\alpha' + I\beta' + (\alpha \vee \beta) \ , \tag{3.10}$$

all cosets of Ω. Again, as mentioned earlier, the above results extend to more general rings such as Von Neumann regular ones. A basic algebraic question that arises here involving (3.9) is how general can a ring Ω be in order for the left hand side of (3.9) to yield a coset of Ω? The answer appears to be tied up with a natural generalization of Von Neumann regularity and is discussed at length in [14].

Next, Corollary 3.1 is a rigorization of a common tacit assumption concerning conditioning, when antecendents are all the same.

Corollary 3.1.

For all $a, b, c \in \Omega$,

$$(a|b) + (c|b) = (a + c|b) = (ab + cb|b) \ , \tag{3.11}$$

$$(a|b) \vee (c|b) = (a \vee c|b) = (ab \vee cb|b) \ , \tag{3.12}$$

$$(a|b) \cdot (c|b) = (ac|b) = (acb|b) \ , \tag{3.13}$$

$$(a|b)' = (a'|b) = (a'b|b) \ , \tag{3.14}$$

with $(0|b)$ and $(1|b) = (b|b)$ playing the roles of, relative to b fixed, additive and multiplicative identities, respectively. □

Remarks.

1. Corollary 3.1 illustrates the reduction of Theorem 3.1, when all antecedents are identical, to the classical natural homomorphism

$$\mathrm{nat}_b : \Omega \to \Omega/b' \ , \tag{3.15}$$

where for any fixed $b \in \Omega$ and all $x \in \Omega$,

$$\mathrm{nat}_b(x) \overset{d}{=} (x|b) = x + \Omega b' \ . \tag{3.16}$$

Equivalently, all of the above extended boolean operations coincide over any Ω/b' with the usual corresponding coset operations.

2. By using the canonical form for binary boolean operators, the closure of complement, conjunction and disjunction certainly implies the closure of all binary and similarly n-ary boolean operators over $\tilde{\Omega}$.

One can extend the partial order $\leq$ over Ω to $\tilde{\Omega}$ through the definition and equivalent property

$$\begin{aligned} (a|b) \leq (c|d) \quad & \text{iff} \quad (a|b) = (a|b) \cdot (c|d) \\ & \text{iff} \quad (c|d) = (c|d) \vee (a|b) \ . \end{aligned} \tag{3.17}$$

This ordering has many properties similar to the partial ordering $\leq$ over Ω (a boolean ring).

Theorem 3.2

The relation $\leq$ over $\tilde{\Omega}$ forms a partial order with the following characterization and meet and join lattice properties for all a, b, c, d, e, f, g, h $\in \Omega$:

$$(a|b) \leq (c|d) \ \underline{\text{iff}}\ ab \leq cd \text{ and } c'd \leq a'b , \tag{3.18}$$

$$(a|b) \leq (c|d),\ (e|f) \ \underline{\text{iff}}\ (a|b) \leq (c|d) \cdot (e|f) , \tag{3.19}$$

$$(a|b) \geq (c|d),\ (e|f) \ \underline{\text{iff}}\ (a|b) \geq (c|d) \vee (e|f) . \tag{3.20}$$

$$\text{If } (a|b) \leq (c|d) \text{ then } (c|d)' \leq (a|b)' . \tag{3.21}$$

$$\text{If } (a|b) \leq (c|d) \text{ and } (e|f) \leq (g|h) \text{, then}$$

$$(a|b) \cdot (e|f) \leq (c|d) \cdot (g|h) , \tag{3.22}$$

$$(a|b) \vee (e|f) \leq (c|d) \vee (g|h) . \tag{3.23}$$

□

The algebraic properties of $\tilde{\Omega}$ are summarized in the following (Ω assumed boolean).

Theorem 3.3.

$\tilde{\Omega}$ relative to $+, \cdot$ is in general not a ring due to failure of additive inverses, though it is commutative and associative for both operations and idempotent for $\cdot$. In addition, relative to $\vee, \cdot$ it is a semi-ring in the algebraic sense (i.e., a semi-group relative to $\vee, \cdot$ and (mutually) distributive) which is commutative with additive identity $0 \in \Omega$ and multiplicative identity (unity) $1 \in \Omega$. In addition, $\tilde{\Omega}$ is DeMorgan for $(\vee,\cdot,(\ )')$ both ways, is mutually absorbing for $\vee$ and $\cdot$ and $(\ )'$ is involutive. □

Some additional miscellaneous properties:

Theorem 3.4.

For all a, b, c, d $\in \Omega$:

$$(a|b) \cdot (a|b') = (a^2|a') = (a|a') = (0|a') , \tag{3.24}$$

$$(a|b) \vee (a|b') = (a|a) = (1|a) , \tag{3.25}$$

$$(a|b) \vee (a|b)' = (b|b) = (1|b) , \tag{3.26}$$

$$(a|b) = a + (0|b) , \tag{3.27}$$

$$c \vee (a|b) = (a \vee c|b \vee c),\ c\cdot(a|b) = (ca|b \vee c') , \tag{3.28}$$

$$c + (a|b) = (c + a|b) , \tag{3.29}$$

$$(a|b) + (c|d) = (a|b)(c|d)' \vee (a|b)'(c|d) , \tag{3.30}$$

$$(a|bc) \cdot (b|c) = (ab|c) \ \text{(chaining property)}. \tag{3.31}$$

For all $a_1, \cdots, a_m \in \Omega$:

If $a_1 \cdots, a_m$ are disjoint and exhaustive, i.e.,

$$a_i a_j = \delta_{ij} \quad \text{and} \quad a_1 + \cdots + a_m = 1, \tag{3.32}$$

then for any j, the Bayes' theorem forms hold:

$$(a_j|b) = (a_j b|b) = ((b|a_j)\cdot a_j|b) \ , \tag{3.33}$$

$$(a_j|b)\cdot b = (b|a_j)\cdot a_j \ (= a_j b) \ , \tag{3.34}$$

$$b = (b|a_1)\cdot a_1 + \cdots + (b|a_m)\cdot a_m \ . \tag{3.35}$$

If $a_1 \le a_2 \le \cdots \le a_m$, then the chaining relation holds:

$$(a_1|a_2)\cdot(a_2|a_3) \cdots (a_{m-1}|a_m) = (a_1|a_m) \ . \tag{3.36}$$

□

Some comparisons between classical logical implication as given in (1.1) and conditioning, with, as usual Ω assumed boolean:

Theorem 3.5.

For all $a, b \in \Omega$,

$$(a|b) = (b \Rightarrow a|b) \ , \tag{3.37}$$

$$a\cdot b, \ b \Rightarrow a \in (a|b) \ , \tag{3.38}$$

$$ab \le (a|b) \le (b \Rightarrow a) \ , \tag{3.39}$$

and in the sense for all $y \in (a|b)$

$$ab \le y \le (b \Rightarrow a) \ , \tag{3.40}$$

$$b \Rightarrow a = (a|b) \vee b' = (a' \Rightarrow b') = (b'|a') \vee a \ , \tag{3.41}$$

$$(a|b) = (b \Rightarrow a) \cdot (b|b) = ((b'|a') \vee a) \cdot (b|b) \tag{3.42}$$

$$(b'|a') = (b \Rightarrow a) \cdot (a'|a') = ((a|b) \vee b') \cdot (a'|a') \tag{3.43}$$

$$(a \Leftrightarrow b) \overset{d}{=} (a \Rightarrow b)\cdot(b \Rightarrow a) = ab \vee a'b' = (a|b)\cdot(b|a) \vee a'b' \tag{3.44}$$

$$(a|b)\cdot(b|a) = (ab|a \vee b) = (a \Leftrightarrow b)\cdot(ab|ab) \ , \tag{3.45}$$

implying, analogous to (3.39),

$$ab \le (a|b)\cdot(b|a) \le (a \Leftrightarrow b). \tag{3.46}$$

□

Remarks.

One can compare the properties of $\Rightarrow$ and $(\cdot|\cdot\cdot)$ side by side and see that certain analogous properties do hold. For example,

$$(a|b) = (ab|b) \ \text{while} \ (b \Rightarrow a) = (b \Rightarrow ab) \ , \tag{3.47}$$

$$(1|b) = (b|b) = \Omega \vee b \ \text{while} \ (b \Rightarrow 1) = (b \Rightarrow b) = 1 \ , \tag{3.48}$$

$$(b|1) = b \ \text{while} \ (1 \Rightarrow b) = b \ , \tag{3.49}$$

$$(b|0) = \Omega \quad \text{while} \quad (0 \Rightarrow b) = 1 \,, \tag{3.50}$$

$$(a|b)' = (a'b|b) \quad \text{while} \quad (b \Rightarrow a)' = a'b \,, \tag{3.51}$$

$$(0|b) = (b'|b) = \Omega\cdot b' \quad \text{while} \quad (b \Rightarrow 0) = (b \Rightarrow b') = b' \,, \tag{3.52}$$

$$(a|b)\cdot(c|d) = (ac|q) \quad \text{while} \quad (b \Rightarrow a)\cdot(d \Rightarrow c) = (q \Rightarrow ac) \,, \tag{3.53}$$

$$(a|b) \vee (c|d) = (a \vee c|r) \quad \text{while} \quad (b \Rightarrow a) \vee (d \Rightarrow c) = (bd \Rightarrow a \vee c) \,, \tag{3.54}$$

where

$$q \overset{d}{=} a'b \vee c'd \vee bd, \quad r \overset{d}{=} ab \vee cd \vee bd \,. \tag{3.55}$$

For $a \leq b \leq c$,

$$(a|b)\cdot(b|c) = (a|c) \quad \text{while} \quad (c \Rightarrow b)\cdot(b \Rightarrow a) \leq (c \Rightarrow a). \tag{3.56}$$

For $a \leq bc$,

$$(a|b) \leq (a|bc) \quad \text{while} \quad (b \Rightarrow a) \leq (b\cdot c \Rightarrow a) \,. \tag{3.57}$$

Finally, this section is concluded with two useful results: the first concerning the intersection of conditional objects generalizing the classical disjoint-identical relation of cosets in a fixed quotient ring, and the second concerning subclass inclusion of conditional objects.

Theorem 3.6.

For any $a, b, c, d \in \Omega$, denoting $\cap$ ordinary component-wise class intersection and $\subseteq$ as the ordinary component-wise subclass relation:

$$\text{(i)} \qquad (a|b) \cap (c|d) = \begin{cases} \emptyset \ \text{ iff } \ a + c \notin (0|bd) \\ (\gamma(a,c)|b \vee d) \ \text{ iff } \ a + c \in (0|bd) \,, \end{cases} \tag{3.58}$$

where

$$\gamma(a,c) \overset{d}{=} p + a = q + c; \; a + c = q + p \,, \tag{3.59}$$

for some $p \in (0|b)$, $q \in (0|d)$.

$$\text{(ii)} \qquad (a|b) \subseteq (c|d) \ \ \underline{\text{iff}} \ \ b \geq d \ \text{ and } \ a \in (c|d) \,. \tag{3.60}$$

□

4. ADDITIONAL PROPERTIES OF CONDITIONAL OBJECTS

Define higher order or iterated conditional objects, analogous to that in (2.3) where $a \in \Omega$ is replaced by $(a|b) \in \tilde{\Omega}$ and $b \in \Omega$ is replaced by $(c|d) \in \tilde{\Omega}$. Thus, for any $a, b, c, d \in \Omega$,

$$((a|b)|(c|d)) = ((a|b)\cdot(c|d)|(c|d)) = f^{-1}_{(c|d)}((a|b)\cdot(c|d))$$

$$= \{(x|y) : (x|y) \in \tilde{\Omega} \ \text{ and } \ (x|y)\cdot(c|d) = (a|b)\cdot(c|d)\} \,, \tag{4.1}$$

the explicit solution of which is given in:

Theorem 4.1.

For all a, b, c, d $\in \Omega$ (boolean)

$$((a|b)|(c|d)) = (a|b)\cdot(c|d) \vee \mathcal{S}_{a,b,c,d}\ , \tag{4.2}$$

$$\mathcal{S}_{a,b,c,d} \overset{d}{=} \{(\beta\cdot t|(c'd)' \vee s)|t \le s \in \Omega\}\ , \tag{4.3}$$

$$\beta = \beta(a,b,c,d) \overset{d}{=} (cd)'(a'bd')' = (a'b)'d' \vee c'd\ , \tag{4.4}$$

$$\beta' = \beta'(a,b,c,d) = cd \vee a'bd' = (c'd)'\cdot(a'b \vee d). \tag{4.5}$$

□

Because of the inherent difficulties in handling higher order conditional objects and possible operator closure problems arising, two approximation procedures have been developed.

The first procedure utilizes the class reduction operator $\bar{U} : \mathcal{P}(\mathcal{P}(\Omega)) \to \mathcal{P}(\Omega)$, where for all $\mathcal{A} \in \mathcal{P}(\mathcal{P}(\Omega))$,

$$\bar{U}(\mathcal{A}) \overset{d}{=} \bigcup_{A\in\mathcal{A}} A = \{x|x \in A \in \mathcal{A}\} \subseteq \Omega\ . \tag{4.6}$$

Denote $\approx{\Omega}$ as the set of second order conditional objects. Then

Theorem 4.2.

$\bar{U} : \overset{\approx}{\Omega} \to \tilde{\Omega}$ is a surjective homomorphism relative to all boolean operations and for all a, b, c, d $\in \Omega$:

$$\bar{U}((a|b)|(c|d)) = (a|\alpha)\ , \tag{4.7}$$

$$\alpha = \alpha(a,b,c,d) \overset{d}{=} b\cdot\beta' = b\cdot(cd \vee a'd')\ . \tag{4.8}$$

Thus,

$$\bar{U}(a|b) = (a|b);\ \bar{U}((a|b)|(c|b)) = \bar{U}((a|b)|c) = (a|bc)\ , \tag{4.9}$$

$$\bar{U}(a|(c|d)) = (a|cd \vee a'd')\ , \tag{4.10}$$

$$(c|d)\cdot\bar{U}((a|b)|(c|d)) = (a|b)\cdot(c|d)\ . \tag{4.11}$$

Also, the following restrictions of $\bar{U}$ are surjective isomorphisms relative to all boolean operations :

$\bar{U} : \{((a|b)|c):a,\ b \in \Omega\} \to \{(a|bc):\ a,\ b \in \Omega\}\ ,$

$\bar{U} : \{((a|b)|(c|b)):a,\ c \in \Omega\} \to \{(a|bc):a,\ c \in \Omega\}\ ,$

$\bar{U} : \{((a|b)|(c|d)):a,\ b \in \Omega\} \to \{(a|\alpha(a,b,c,d)):a,\ b \in \Omega\}\ .$

□

Thus in a real sense $\bar{U}$ reduces all higher order conditional objects and their operations down to $\tilde{\Omega}$.

The thrust of the second procedure is to determine for a given $\mathcal{A} \in \mathcal{P}(\mathcal{P}(\Omega))$, the best upper approximation by $\approx\Omega$, i.e., the smallest possible element $(a|b)|(c|d) \in \approx\Omega$ such that $\mathcal{A} \subseteq ((a|b)|(c|d))$, relative to component-wise subclass inclusion. Since the analogue of this procedure for $\tilde{\Omega}$ is of some importance, consider the following result utilizing Theorem 3.6:

Theorem 4.3.

If Ω is not only boolean but complete, i.e., $\vee$ and $\cdot$ are extendable over Ω to any infinitude of multiple arguments, where one defines for any $A \subseteq \Omega$, $\vee(A) \overset{d}{=} \underset{x \in A}{\vee}(x)$, etc. Then for any $A \subseteq \Omega$, denoting $\cdot$ as $\wedge$,

$$\mathcal{F}(A) \overset{d}{=} \{(a|b) | A \subseteq (a|b) \in \tilde{\Omega}\}$$

$$= \{(a|b):(a|b) \in \tilde{\Omega},\ \wedge(A) \vee (v(A))' \geq b,\ \wedge(A) \in (a|b)\}\ , \qquad (4.12)$$

$$A \subseteq \mathrm{cond}(A) \overset{d}{=} \cap\ \mathcal{F}(A) = (\wedge(A)|\wedge(A) \vee (v(A))') \in \tilde{\Omega}\ . \qquad (4.13)$$

□

Corollary 4.1.

Let $f : \Omega \times \Omega \to \Omega$ be increasing, i.e., for all $a_i, b_i \in \Omega$, $i = 1, 2$

$$a_1 \leq a_2,\ b_1 \leq b_2 \ \text{ implies } \ f(a_1,b_1) \leq f(a_2,b_2)\ . \qquad (4.14)$$

Then extending f to $f : \tilde{\Omega} \times \tilde{\Omega} \to \mathcal{P}(\Omega)$, one need not have $\mathrm{rng}(f) \subseteq \tilde{\Omega}$. However, for all $(a|b)$, $(c|d) \in \tilde{\Omega}$,

$$\mathrm{cond}(f((a|b),\ (c|d))) = (f(a,c)|f(a,c) \vee (f(a \vee b', c \vee d'))')\ . \qquad (4.15)$$

□

Corollary 4.2.

Let $\Omega = \mathcal{B}^n$, the real borel σ-ring of subsets of $\mathbb{R}^n$ and let $f : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ be any function. Let $f : \Omega \times \Omega \to \Omega$ be the usual component-wise class extension of f. Then f over $\tilde{\Omega} \times \tilde{\Omega}$ is always increasing, where $\leq$ is interpreted here as $\subseteq$ between sets in Ω, and hence (4.15) is valid.

In particular, arithmetic operations over $\mathbb{R}^n \times \mathbb{R}^n$ such as ordinary addition and multiplication in general can be shown not always to have ranges in $\tilde{\Omega}$, for $\Omega = \mathcal{B}^n$. Thus, Corollary 4.2 is useful in making the modified extension. Using the fact that for any $\lambda \in \mathbb{R}$ and any $a \subseteq b \in \mathcal{B}^n$,

$$\lambda \cdot (a|b) = (\lambda a|\lambda b) , \tag{4.16}$$

it follows for any $\lambda_i \in \mathbb{R}$ with $0 \le \lambda_i \le 1$. $i = 1, \cdots, m$, $\lambda_1 + \cdots + \lambda_m = 1$, where here $\cdot$, $+$, 1 refer to ordinary multiplication, sum and unity, respectively, and for any $a_i \subseteq b_i \in \mathcal{R}^n$, $i = 1, \cdots, m$,

$$\mathrm{cond}\left(\sum_{i=1}^{m} \lambda_i \cdot (a_i|b_i) \right) = \left(\sum_{i=1}^{m} \lambda_i a_i \,\middle|\, \sum_{i=1}^{m} \lambda_i a_i \cup K'_{a,b} \right), \tag{4.17}$$

where

$$K_{a,b} \overset{d}{=} \sum_{i=1}^{m} \lambda_i \cdot (a_i \cup b'_i) = \bigcup_{J \subseteq I_m} \left(\sum_{i \in J} \lambda_i a_i + \sum_{j \in I_m - J} \lambda_j b'_j \right) . \tag{4.18}$$

Returning to $\tilde{\tilde{\Omega}}$, with Ω again a general boolean ring, in considering the cond( ) operator, the analogue of (3.60) for $\tilde{\tilde{\Omega}}$ is important.

Theorem 4.4.

For all $a_i, b_i, c_i, d_i \in \Omega$ with $(a_i|b_i) \le (c_i|d_i)$ and $a_i \le b_i$, $c_i \le d_i$, without loss of generality, $i = 1, 2$,

$$((a_1|b_1)|(c_1|d_1)) \subseteq ((a_2|b_2)|(c_2|d_2)) \tag{4.19}$$

iff

$$a_2 = c_o a_1 , \quad b_2 = d_o \cdot (c'_o \vee b_1 \vee d'_1) \vee a'_1 b_1 \tag{4.20}$$

$$c_2 = c_o \beta'_1 , \quad d_2 = d_o \vee c'_1 d_1 , \tag{4.21}$$

where $c_o, d_o \in \Omega$ are arbitrary with

$$c_2 \le d_o , \tag{4.22}$$

and

$$\beta_1 \overset{d}{=} b'_i d'_i \vee c'_i d_i , \quad \beta'_i = c_i \vee b_i d'_i , \tag{4.23}$$

$i = 1, 2$. □

Use of Theorem 4.4 enables cond to be obtained for disjunction and complement of objects in $\tilde{\tilde{\Omega}}$. Then, noting $\mathrm{cond}(\mathcal{A}) = \mathcal{A}$, if $\mathcal{A} \in \tilde{\tilde{\Omega}}$, for cond operating over $\mathcal{P}(\mathcal{P}(\Omega))$:

Theorem 4.5.

For $a_i, b_i, c_i\ d_i \in \Omega$ with $(a_i|b_i) \le (c_i|d_i)$, $a_i \le b_i$, $c_i \le d_i$, $i = 1, 2$:

$$((a_1|b_1)|(c_1|d_1)) \vee ((a_2|b_2)|(c_2|d_2)) = ((a_3|b_3)|(c_3|d_3)) \ , \tag{4.24}$$

where

$$a_3 \overset{d}{=} a_1 \vee a_2,\ b_3 \overset{d}{=} a_3 \vee b_1 b_2 \ , \tag{4.25}$$

$$c_3 \overset{d}{=} a_3 \vee \beta_1' \beta_2',\ d_3 \overset{d}{=} c_3 \vee b_1 b_2 \ , \tag{4.26}$$

but

$$((a_1|b_1)|(c_1|d_1))' \subset \mathrm{cond}(((a_1|b_1)|(c_1|d_1))')$$
$$= ((a_4|b_4)|c_4) \ , \tag{4.27}$$

where

$$a_4 \overset{d}{=} \beta_1' a_1' b_1 \ , \quad b_4 \overset{d}{=} b_1 \vee \beta_1 \ , \quad c_4 \overset{d}{=} \beta_1' \ , \tag{4.28}$$

so that $((a_1|b_1)|(c_1|d_1))' \notin \tilde{\tilde{\Omega}}$, in general. □

Finally, it can be shown that Stone's Representation Theorem can be extended to an order preserving isomorphism between $\tilde{\Omega}$ and a corresponding conditional set space by simply using the extension of the initial isomorphism ψ in Stone's Theorem linking any abstract boolean ring Ω with a boolean ring of sets defined by $\psi(a|b) = (\psi(a)|\psi(b))$, for all $a, b \in \Omega$.

5. SOME RELATIONS BETWEEN CONDITIONAL OBJECTS AND CONDITIONAL PROBABILITIES

So far, conditional objects have been developed without regard to any particular probability measure. But if the concept is to make sense, compatible links must be established with conditional probabilities.

Let $p : \Omega \to [0,1]$ be a probability measure with Ω a boolean ring. Extend p to $p : \tilde{\Omega} \to [0,1]$, noting (2.12), (2.14). Although for each $b \in \Omega$, $p_b : \Omega \to [0,1]$ or equivalently $p_b : \Omega b \to [0,1]$ is a probability measure, where p_b is the conditional probability measure

$$p_b(a) \overset{d}{=} p(a|b) = p(ab)/p(b) \ , \tag{5.1}$$

$p : \tilde{\Omega} \to [0.1]$ is neither additive nor subadditive for conditional objects with different antecedents. However, $p : \tilde{\Omega} \to [0,1]$ can be shown to be monotone increasing, i.e., if $(a|b) \le (c|d) \in \tilde{\Omega}$, then

$$p(a|b) \le p(c|d) \ . \tag{5.2}$$

Applying (5.2) to (3.39), e.g., leads to

$$p(ab) \le p(a|b) \le p(b \Rightarrow a) \ , \tag{5.3}$$

for all $a, b \in \Omega$, verifying, as a check, (1.2). Note also that Theorem 3.1

shows p can be evaluated on any boolean function of conditional objects as a simple conditional probability.

In another direction, define two conditional objects $(a|b), (c|d) \in \tilde{\Omega}$ to be qualitatively or measure-free independent iff they are p-independent, i.e.,

$$p((a|b)\cdot(c|d)) = p(a|b)\cdot p(c|d) \ , \qquad (5.4)$$

for all probability measures $p : \Omega \to [0,1]$. Similarly, (3.36) shows, for $a_1 \leq \cdots \leq a_m$, $(a_1|a_2), \cdots, (a_{m-1}|a_m)$ are jointly qualitatively independent. (See also Domotor's somewhat different concept of qualitative independence ([15], Section 2.5).)

<u>Theorem 5.1</u>.

The only examples of qualitative independent pairs of conditional objects $(a|b), (c|d) \in \tilde{\Omega}$ are when

$$(a \leq b = c \leq d), \text{ or } (a \leq b, c; c = d) \ , \qquad (5.5)$$

or

$$(a = b, c = d), \text{ or } c = 0 \ . \qquad (5.6)$$

□

Another direction of interest involves sequential updating of information. If a represents an event of interest and $b, c, d, \cdots$ are *successively arriving observed data events, then identifying* $(a|b\cdot c)$ with $(a|b,c)$, etc., and utilizing (4.9), one obtains

$$p(\bar{U}((a|b)|(c|b))) = p_{bc}(a) = p_c(a|b)$$

$$= p((a|b)\cdot(c|b))/p(c|b) \overset{d}{=} p((a|b)|(c|b)) \ . \qquad (5.7)$$

By replacing in (5.7) b by $b\cdot c$ and c by d, one obtains

$$p(\bar{U}((a|b,c)|(d|b,c))) = p_{bcd}(a) = p_d(a|b,c)$$

$$= p((a|b,c)\cdot(d|b,c))/p(d|b,c) = p((a|b,c)|(d|b,c)) \ . \qquad (5.8)$$

Thus one can substitute the conditional forms in (5.7) into (5.8) iteratively, with in effect higher order conditional objects generating the desired updating.

Finally, we consider briefly the randomization of conditional objects and their relation to ordinary conditional random variables.

Let $(M,\mathcal{A},p)$ be a probability space and $V : M \to \mathbb{R}^m$, $W : M \to \mathbb{R}^n$ random variables with measurable ranges. Extend V and W by the component-wise class procedure up to $\tilde{V} : \tilde{\mathcal{A}} \to \tilde{\mathcal{B}}^m$, $\tilde{W} : \tilde{\mathcal{A}} \to \tilde{\mathcal{B}}^n$ and $V \times W$ to $\widetilde{V \times W} : \tilde{\mathcal{A}} \to \tilde{\mathcal{B}}^{m+n}$, where for all $a, b \in \mathcal{A}$,

$$V(a|b) \overset{d}{=} V(a) \times \mathbb{R}^n \ , \quad W(a|b) \overset{d}{=} \mathbb{R}^m \times W(b) \ ,$$

$$(V \times W)(a|b) \overset{d}{=} V(a) \times W(b) \ . \tag{5.9}$$

Next, define the random conditional object $(V|W) : \tilde{\mathcal{A}} \to \tilde{\mathcal{B}}^{m+n}$ by, for all a, $b \in \mathcal{A}$

$$(V|W)(a|b) \overset{d}{=} (V \times W)(a|b)|W(a|b))$$

$$= (V(a) \times W(b)|\mathbb{R}^m \times W(b)) \overset{d}{=} (V(a)|W(b)) \ , \tag{5.10}$$

whence the inverse mapping $(V|W)^{-1} : \tilde{\mathcal{B}}^{m+n} \to \tilde{\mathcal{A}}$ yields for any $c \in \mathcal{B}^m$, $d \in \mathcal{B}^n$,

$$(V|W)^{-1}(c|d) = ((V \times W)^{-1}(c \times d)|W^{-1}(d))$$

$$= (V^{-1}(c)|W^{-1}(d)) \ . \tag{5.11}$$

Hence $(V|W)$ induces "conditional event probability space" $(\mathbb{R}^{m+n}, \tilde{\mathcal{B}}^{m+n}, p_{(V|W)})$, where $p_{(V|W)} : \tilde{\mathcal{B}}^{m+n} \to [0,1]$ is given by

$$p_{(V|W)}(c|d) = p((V|W) \text{ is in } (c|d))$$

$$= p((V|W)^{-1}(c|d))$$

$$= p(V^{-1}(c)|W^{-1}(d)) \ . \tag{5.12}$$

Finally, one can define, e.g., the expectation of random conditional objects as a limit of refined partitions of weighted sums of conditional objects with weights in the form of (5.12) for c, d replaced by suitable c_i, d_i from the partitions,. Thus, using (4.17), (4.18), one can show

$$E((V|W \in d)) = (E \langle V|W \in d\rangle|d) \ , \tag{5.13}$$

$$E((V|W)) = (E \langle V \times W\rangle|E\langle V \times W\rangle) \ , \tag{5.14}$$

where $E\langle\cdot|\cdot\cdot\rangle$ indicates ordinary conditional expectation.

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